A Sequence of Coin Toss Variables for which the Strong Law Fails

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Introduction.

Let $X = \{X_1, X_2, \cdots\}$ be a sequence of $\{0,1\}$-valued random variables defined on a probability space $(\Omega, \mathcal{F}, P)$. Call X a coin toss sequence if

$$(1) \qquad P[X_1 = i_1,\ X_2 = i_2, \cdots,\ X_k = i_k] = 2^{-k}$$

for every finite sequence $i_1, \cdots, i_k$ of 0's and 1's. It follows easily from (1) that $X_1, X_2, \cdots$ are independent and identically distributed.

Define $S_n = X_1 + \cdots + X_n$ for $n = 1, 2, \cdots$. The weak and the strong laws of large numbers for the sequence X are, respectively,

$$(2) \qquad \lim_{n \to \infty} P[|S_n/n - 1/2| \leq \varepsilon] = 1 \quad \text{for every } \varepsilon > 0.$$

and

$$(3) \qquad P[\lim_{n \to \infty} S_n/n = 1/2] = 1.$$

Both (2) and (3) are classical results if $(\Omega, \mathcal{F}, P)$ is a conventional countably additive probability space in which $\mathcal{F}$ is a σ-field of subsets of Ω and P is a countably additive probability measure on $\mathcal{F}$.

Suppose instead that $(\Omega, \mathcal{F}, P)$ is a finitely additive probability space in which $\mathcal{F}$ is a field of subsets of Ω and P is a finitely additive probability measure on $\mathcal{F}$. The weak law remains true. Indeed the usual proof that (1) implies (2) does not rely on countable additivity. However, it is well-known to students of finite additivity that the strong law need not hold.

At least two recent papers have given examples in which the strong law fails (Dubins and Freedman [2], Kumar and Fine [5]). The examples presented in these papers are non-constructive and rely on more axioms than those of ZF (ZF is the Zermelo-Fraenkel set theory without the axiom of choice). The example presented in the next section is completely natural and constructive. It has the property that the convergence set $C \equiv [\lim_{n\to\infty} S_n/n = 1/2]$ is empty so that no extension of P is necessary to see that (3) fails. A slight refinement gives a constructive example for which the set $D \equiv [S_n/n \text{ converges}]$ is empty. Thus finitely additive spaces can be used to model bounded, stationary sequences for which averages fail to converge. Real world examples of such phenomena have been reported by Kumar and Fine [5] who suggest non-additive models.

The Example.

Let $\Omega = \{1,2,\cdots\}$ be the set of positive integers and let $\mathcal{F}$ be the collection of all finite unions of congruence classes of the form

$$A(n,r) = \{\omega \in \Omega \mid \omega \text{ had remainder } r \text{ when divided by } n\}$$

where $n = 1,2,\cdots;\ r = 0,1,\cdots, n-1$. It is easy to check that $\mathcal{F}$ is an algebra.

It is also straightforward to verify that there is a unique finitely additive probability measure P on $\mathcal{F}$ satisfying

$$P(A(n,r)) = 1/n \quad \text{for every set } A(n,r). \tag{4}$$

The space $(\Omega,\mathcal{F},P)$ is familiar to number theorists who consider certain extensions of P. It was also mentioned by Dubins and Savage [3] in their seminal book on finite additivity and gambling.

To define the coin toss sequence $X = \{X_1,X_2,\cdots\}$, write $\omega \in \Omega$ in its unique binary expansion

$$\omega = a_1 \times 2^0 + a_2 \times 2^1 + \cdots$$

and set

$$X_k(\omega) = a_k.$$

For a sequence $i_1, \cdots, i_k$ of 0's and 1's let

$$i_k \cdots i_1 = i_1 \times 2^0 + \cdots + i_k \times 2^{k-1}.$$

It can then be verified that

$$\{\omega | X_1(\omega) = i_1, \cdots, X_k(\omega) = i_k\} = A(2^k, i_k \cdots i_1). \tag{5}$$

(For example, $[X_1 = 0]$ is the set of even numbers.) Property (1) now follows from (4) and (5). Thus X is a coin toss sequence.

Now, for every $\omega \in \Omega$, there is a positive integer k_0 such that $a_k = X_k(\omega) = 0$ for $k \geq k_0$. Hence, $S_n(\omega)/n$ converges to 0 for every ω.

Next we will construct a coin toss sequence $Y = \{Y_1, Y_2, \cdots\}$ such that $(Y_1(\omega) + \cdots + Y_n(\omega))/n$ fails to converge, for every ω. To define Y, let $j_1, j_2, \cdots$ be an infinite sequence of 0's and 1's such that $(j_1 + \cdots + j_n)/n$ does not converge. Let

$$Y_n = X_n \qquad \text{if } j_n = 0,$$
$$= 1 - X_n \quad \text{if } j_n = 1.$$

It is easily checked that Y is a coin toss sequence. Also, for every ω, $Y_n(\omega)$ is eventually equal to j_n because $X_n(\omega)$ is eventually equal to 0.

By the way, an example quite similar to the above can be constructed by taking Ω to be the rational numbers in [0,1], $\mathcal{F}$ to be all finite unions of intervals of rationals and P to be the unique probability on $\mathcal{F}$ such that P(I) = "length" of I for every interval I of rationals. The random variables $X_1, X_2, \cdots$ correspond to a binary expansion as before.

Remarks.

We have shown this example to a number of our friends who are conventional, countably additive probabilists. Most of them are amused by the example and consider it to be evidence of the perversities of finite additivity. However, we view it as evidence of the arbitrariness of the assumption of countable additivity since we know of no simpler or more elementary model of a coin toss sequence. It seems to us a mistake to ban such an example from consideration. (A recent editor of the *Annals of Probability* ruled that "finitely additive probability is not probability.")

There is a natural class of finitely additive probability measures which includes the countably additive measures and for which the classical strong laws of probability theory do hold. (See Purves and Sudderth [6], Chen [1], Ramakrishnan [7] and Karandikar [4].)

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